A RECIPROCITY AND FOUR-TERM RELATION FOR GENERALIZED DEDEKIND SUMS

BY

L. CARLITZ

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1. Introduction

Let $B_n(x)$ denote the polynomial of degree n defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \tag{1.0}$$

and let $\bar{B}_n(x)$ denote the Bernoulli function:

$$\bar{B}_n(x) = B_n(x-[x]), \tag{1.1}$$

where $[x]$ is the greatest integer $\leqslant x$. By (1.1), $\bar{B}_n(x)$ is periodic with period 1. Now put

$$\phi_{r,s}(h, k) = \sum_{a \,(\mathrm{mod}\, k)} \bar{B}_r\left(\frac{ha}{k}\right)\bar{B}_s\left(\frac{a}{k}\right), \tag{1.2}$$

where r, s are arbitrary non-negative integers and the summation is over a complete residue system (mod k). Also define

$$\psi_{r,s}(h, k) = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} h^{r-j}\, \phi_{j,\, r+s-j}\,(h, k). \tag{1.3}$$

It follows from (1.2) that

$$\phi_{r,s}(h+k, k) = \phi_{r,s}(h, k). \tag{1.4}$$

For $\psi_{r,s}(h, k)$, however, we have

$$\psi_{r,s}\,(h+k, k) = \sum_{t=0}^{r} (-1)^t \binom{r}{t} k^t\, \psi_{r-t,\, s+t}\,(h, k). \tag{1.5}$$

It is also clear from (1.2) that, if $(h, k)=1$ and $hh' \equiv 1 \pmod{k}$, then

$$\phi_{r,s}(h, k) = \phi_{s,r}(h', k). \tag{1.6}$$

The corresponding formula for $\psi_{r,s}(h, k)$ is

$$\psi^*_{r,s}(h, k) = \psi_{s,r}(h, k), \tag{1.7}$$

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where

$$(1.8) \qquad \psi^*_{r,s}(h,k) = \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} h^{s-j} \phi_{r+s-j,j}(h',k).$$

By making use of a transformation formula for the function

$$(1.9) \qquad G_p(x) = \sum_{m,n=1}^{\alpha} n^{-p} x^{mn} \qquad (|x|<1)$$

proved by APOSTOL [1, Th. 2], the writer has proved [5, Th. 1] the following reciprocity theorem:

$$(1.10) \quad \begin{cases} \binom{r+s}{r} k^r \psi_{s,r}(h,k) - \binom{r+s}{r} kB_rB_s \\ \qquad = \binom{r+s}{r+1} h^{s-1} \psi_{r+1,s-1}(k,h) - \binom{r+s}{r+1} hB_{r+1}B_{s-1}, \end{cases}$$

where $(h,k)=1$, $r+s$ is even, $r+s>2$, $r \geqslant 0$, $s>0$, and $B_r = B_r(0)$. For $s=1$, (1.10) reduces to APOSTOL's reciprocity theorem [1, Th. 2; 2, Th. 2].

RADEMACHER [7] has proved a three-term relation for the ordinary Dedekind sum

$$(1.11) \qquad s(h,k) = \sum_{t=1}^{k-1} \frac{t}{k}\left(\frac{ht}{k} - \left[\frac{ht}{k}\right] - \frac{1}{2}\right),$$

namely

$$(1.12) \qquad s(bc',a) + s(ca',b) + s(ab',c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right),$$

where a, b, c are relatively prime in pairs and

$$aa' \equiv 1 \pmod{bc}, \quad bb' \equiv 1 \pmod{ca}, \quad cc' \equiv 1 \pmod{ab}.$$

The writer [4] has obtained linear relations for the generalized sums $\phi_{r,s}(h,k)$ that include (1.12) as a special case.

In the present paper we first give an elementary proof of (1.10) which incidentally removes the restrictions on r, s. The proof is along the lines of the author's proof [6] of the reciprocity theorem for the simple DEDEKIND sum [8]. We then give a simple proof, by the same method, of a four-term relation, which is essentially an improved form of the result in [4] mentioned above. It may be of interest to mention that the proof depends upon the following identity:

$$ac \sum_{\eta} \frac{u^c \eta^{c-a}}{(u^a - \eta^{-a})(v^c - u^c \eta^c)} + ab \sum_{\zeta} \frac{v^b \zeta^{b-a}}{(v^a - \zeta^{-a})(u^b - v^b \zeta^b)}$$
$$= bc \sum_{\xi} \frac{\xi^{b+c}}{(u^b - \xi^b)(v^c - \xi^c)},$$

where $(b, c)=(c, a)=(a, b)=1$ and ξ, η, ζ run through the roots of unity of index a, b, c, respectively.

2. Let $(h, k)=1$. We shall require the following identity:

$$k \sum_{\eta} \frac{u^k \eta^{k-1}}{(u-\eta^{-1})(v^k-u^k\eta^k)} + h \sum_{\zeta} \frac{v^h \zeta^{h-1}}{(v-\zeta^{-1})(u^h-v^h\zeta^h)} = \frac{hk}{(u^h-1)(v^k-1)}, \tag{2.1}$$

where the first summation is over all k-th roots of unity and the second is over all h-th roots of unity.

PROOF. It follows from

$$u^k-1=\prod_{\zeta}(u-\zeta)$$

that

$$\frac{ku^k}{u^k-1}=\sum_{\zeta}\frac{u}{u-\zeta}. \tag{2.2}$$

Replacing u by $u\eta/v$, this becomes

$$\frac{ku^k\eta^k}{u^k\eta^k-v^k}=\sum_{\zeta}\frac{u\eta}{u\eta-v\zeta}. \tag{2.3}$$

Similarly

$$\frac{hv^h\zeta^h}{v^h\zeta^h-u^h}=\sum_{\eta}\frac{v\zeta}{v\zeta-u\eta}. \tag{2.4}$$

By (2.3)

$$\sum_{\eta}\frac{ku^k\eta^{k-1}}{(u-\eta^{-1})(v^k-u^k\eta^k)}=\sum_{\eta}\frac{\eta^{-1}}{u-\eta^{-1}}\frac{u\eta}{v\zeta-u\eta}$$

and by (2.4)

$$\sum_{\zeta}\frac{hv^h\zeta^{h-1}}{(v-\zeta^{-1})(u^h-v^h\zeta^h)}=\sum_{\zeta}\frac{-1}{v-\zeta^{-1}}\sum_{\eta}\frac{v\zeta}{u\eta-v\zeta}.$$

Hence the left hand side of (2.1) is equal to

$$\sum_{\eta,\zeta}\frac{1}{u\eta-v\zeta}\left(\frac{v}{v-\zeta^{-1}}-\frac{u}{u-\eta^{-1}}\right)=\sum_{\eta,\zeta}\frac{1}{u\eta-v\zeta}\frac{u\zeta^{-1}-v\eta^{-1}}{(u-\eta^{-1})(v-\zeta^{-1})}$$
$$=\sum_{\eta,\zeta}\frac{\eta^{-1}\zeta^{-1}}{(u-\eta^{-1})(v-\zeta^{-1})}=\sum_{\eta}\frac{\eta}{u-\eta}\sum_{\zeta}\frac{\zeta}{v-\zeta}.$$

Since, by (2.2),

$$\sum_{\zeta}\frac{\zeta}{v-\zeta}=\frac{1}{v^k-1},$$

it is clear that we have proved (2.1).

3. In (2.1) replace u by e^x and v by e^y. We get

$$
(3.1)\qquad \left\{\begin{aligned} & h \sum_{\zeta} \frac{\zeta^{h-1}}{(e^y-\zeta^{-1})(e^{h(x-y)}-\zeta^h)} + k \sum_{\eta} \frac{\eta^{k-1}}{(e^x-\eta^1)(e^{k-(y-x)}-\eta^k)} \\ & \quad = \frac{hk}{(e^{hx}-1)(e^{ky}-1)} . \end{aligned}\right.
$$

It will be convenient to consider the terms for $\zeta=1$, $\eta=1$ separately. Also we replace x by x/h and y by y/k. Thus (3.1) becomes, after dividing both sides by hk and multiplying by xy,

$$
(3.2)\qquad \left\{\begin{aligned} & \frac{1}{k}\frac{xy}{(e^{y/k}-1)(e^{x-hy/k}-1)} + \frac{1}{h}\frac{xy}{(e^{x/h}-1)(e^{y-kx/h}-1)} \\ & \qquad + \frac{xy}{k} \sum_{\eta \neq 1} \frac{\zeta^{h-1}}{(e^{y/k}-\zeta^{-1})(e^{x-hy/k}-\zeta^h)} \\ & \qquad + \frac{xy}{h} \sum_{\eta \neq 1} \frac{\eta^{k-1}}{(e^{x/h}-\eta^{-1})(e^{y-kx/h}-\eta^k)} = \frac{xy}{(e^x-1)(e^y-1)} . \end{aligned}\right.
$$

By (1.0) we have

$$
(3.3)\qquad \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x-1} .
$$

Also we define the Eulerian function $H_n(\lambda)$ by means of [3]

$$
(3.4)\qquad \frac{1-\lambda}{e^x-\lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{x^n}{n!} \quad (\lambda \neq 1).
$$

It is proved in [4] that

$$
(3.5)\qquad \phi_{r,s}(h,k) = \frac{B_r\, B_s}{k^{r+s-1}} + \frac{rs}{k^{r+s-1}} \sum_{\zeta \neq 1} \frac{H_{r-1}(\zeta^{-1}) H_{s-1}(\zeta^h)}{(\zeta-1)(\zeta^{-h}-1)} ,
$$

where $r \geqslant 1$, $s \geqslant 1$. However, if we define $H_{-1}(\lambda)=0$, it is easily verified that (3.5) holds for all non-negative r, s.

Put

$$
(3.6)\qquad \Phi(h,k|x,y) = \sum_{r,s=0}^{\infty} \phi_{r,s}(h,k) \frac{x^r y^s}{r!s!} .
$$

Then by (3.3), (3.4) and (3.5),

$$
\begin{aligned} \Phi(h,k|kx,ky) &= \frac{kxy}{(e^x-1)(e^y-1)} + kxy \sum_{\zeta \neq 1} \frac{1}{(\zeta-1)(\zeta^{-h}-1)} \sum_{r,s} H_r(\zeta^{-1}) H_s(\zeta^h) \\ & \quad \cdot \frac{x^r y^s}{r!s!} = \frac{kxy}{(e^x-1)(e^y-1)} + kxy \sum_{\zeta \neq 1} \frac{1}{(\zeta-1)(\zeta^{-h}-1)} \frac{1-\zeta^{-1}}{e^x-\zeta^{-1}} \frac{1-\zeta^h}{e^y-\zeta^h} , \end{aligned}
$$

so that

$$
(3.7)\qquad \Phi(h,k|kx,ky) = \frac{kxy}{(e^x-1)(e^y-1)} + kxy \sum_{\zeta \neq 1} \frac{\zeta^{h-1}}{(e^x-\zeta^{-1})(e^y-\zeta^h)} .
$$

Next, if we put

$$\Psi(h, k|x, y)=\sum_{r,s=0}^{\infty} \psi_{r,s}(h, k)\frac{x^r y^s}{r!s!}, \tag{3.8}$$

it follows readily from (1.3) that

$$\Psi(h, k|x, y)=\Phi(h, k|x, y-hx). \tag{3.9}$$

Thus, by (3.7), we have

$$\Psi(h, k|kx, ky)=k\frac{x}{e^x-1}\frac{y-hx}{e^{y-hx}-1}+kx(y-hx)\sum_{\zeta\neq 1}\frac{\zeta^{h-1}}{(e^x-\zeta^{-1})(e^{y-hx}-\zeta^h)}.$$

Changing the notation slightly, this becomes

$$\left\{\begin{aligned}\Psi(h, k|y, kx)&=\frac{1}{k}\frac{y}{e^{y/k}-1}\frac{kx-hy}{e^{x-hy/k}-1}\\ &\quad+\frac{1}{k}y(kx-hy)\sum_{\zeta\neq 1}\frac{\zeta^{h-1}}{(e^{y/k}-\zeta^{-1})(e^{x-hy/k}-\zeta^h)}.\end{aligned}\right. \tag{3.10}$$

Similarly, we have

$$\left\{\begin{aligned}\Psi(k, h|x, hy)&=\frac{1}{h}\frac{x}{e^{x/h}-1}\frac{hy-kx}{e^{y-kx/h}-1}\\ &\quad+\frac{1}{h}x(hy-kx)\sum_{\eta\neq 1}\frac{\eta^{k-1}}{(e^{x/h}-\eta^{-1})(e^{y-kx/h}-\eta^k)}.\end{aligned}\right. \tag{3.11}$$

Thus

$$\begin{aligned}&x\Psi(h, k|y, kx)-y\Psi(k, h|x, hy)\\ &\qquad=xy(kx-hy)\left\{\frac{1}{k}\frac{xy}{(e^{y/k}-1)(e^{x-hy/k}-1)}+\frac{1}{h}\frac{xy}{(e^{x/h}-1)(e^{y-kx/h}-1)}\right.\\ &\qquad+\frac{xy}{k}\sum_{\zeta\neq 1}\frac{\zeta^{h-1}}{(e^{y/k}-\zeta^{-1})(e^{x-hy/k}-\zeta^h)}\\ &\qquad\left.+\frac{xy}{h}\sum_{\eta\neq 1}\frac{\eta^{k-1}}{(e^{x/h}-\eta^{-1})(e^{y-kx/h}-\eta^k)}\right\}.\end{aligned}$$

Hence, by (3.2), we have

$$x\Psi(h, k|y, kx)-y\Psi(k, h|x, hy)=(kx-hy)\frac{xy}{(e^x-1)(e^y-1)}. \tag{3.12}$$

4. Since, by (3.8),

$$\Psi(h, k|y, kx)=\sum_{r,s=0}^{\infty}\psi_{s,r}(h, k)\frac{(kx)^r y^s}{r!\,s!},$$

$$\Psi(k, h|x, hy)=\sum_{r,s=1}^{\infty}\psi_{r,s}(k, h)\frac{x^r(hy)^s}{r!\,s!},$$

(3.12) yields the identity

$$(4.1)\quad \left\{\begin{aligned} &x \sum_{r,s=1}^{\infty} \psi_{s,r}(h,k)\frac{(kx)^r y^s}{r!\,s!} - y \sum_{r,s=0}^{\infty} \psi_{r,s}(k,h)\frac{x^r (hy)^s}{r!\,s!} \\ &\qquad = (kx-hy)\sum_{r,s=0}^{\infty} B_r B_s \frac{x^r y^s}{r!\,s!}. \end{aligned}\right.$$

Comparing coefficients of $x^r y^s$ on both sides of (4.1) we get

$$(4.2)\quad rk^{r-1}\psi_{s,r-1}(h,k) - sh^{s-1}\psi_{r,s-1}(k,h) = rkB_{r-1}B_s - shB_rB_{s-1} \quad (r \geqslant 1,\ s \geqslant 1).$$

If we put $B_{-1}=0$ and $\psi_{r,-1}=0$, then (4.2) holds for all non-negative r, s. For example, if $s=0$, (4.2) becomes

$$rk^{r-1}\psi_{0,r-1}(h,k) = rkB_{r-1} \quad (r \geqslant 1)$$

or, if we prefer

$$(4.3)\qquad k^r \psi_{0,r}(h,k) = kB_r \quad (r \geqslant 0).$$

By (1.2) and (1.3),

$$\psi_{0,r}(h,k) = \phi_{0,r}(h,k) = \sum_{a \pmod k} \bar{B}_r\left(\frac{a}{k}\right) = k^{1-r}\,\bar{B}_r(0) = k^{1-r}\,B_r,$$

by the multiplication theorem for $\bar{B}_r(x)$.

We may now state the following result.

THEOREM 1. *Let $(h,k)=1$ and let r, s be arbitrary non-negative integers. Then we have*

$$(4.4)\qquad rk^{r-1}\,\psi_{s,r-1}(h,k) - sh^{s-1}\,\psi_{r,s-1}(k,h) = rkB_{r-1}\,B_s - shB_rB_{s-1}.$$

As pointed out in the Introduction, it is unnecessary to make any assumption about the parity of $r+s$.

If $r+s$ is odd, the functions $\phi_{r,s}(h,k)$, $\psi_{r,s}(h,k)$ simplify considerably. Indeed it follows from the formulas

$$\left\{\begin{array}{ll} \bar{B}_n(-x) = (-1)^n \bar{B}_n(x) & (n \neq 1) \\ \bar{B}_1(-x) = -\bar{B}_1(x) & (x \neq \text{integer}) \end{array}\right.$$

that

$$\begin{aligned} \phi_{r,s}(h,k) &= \sum_{a \pmod k} \bar{B}_r\left(\frac{ha}{k}\right)\bar{B}_s\left(\frac{a}{k}\right) = \sum_{a \pmod k} \bar{B}_r\left(-\frac{ha}{k}\right)\bar{B}_s\left(-\frac{a}{k}\right) \\ &= B_r\,B_s + \sum_{a=1}^{k-1} (-1)^{r+s}\,\bar{B}_r\left(\frac{ha}{k}\right)\bar{B}_s\left(\frac{a}{k}\right) \\ &= (1-(-1)^{r+s})B_r\,B_s + (-1)^{r+s}\,\phi_{r,s}(h,k). \end{aligned}$$

Therefore

$$(4.5)\qquad \phi_{r,s}(h,k) = B_rB_s \quad (r+s \text{ odd}).$$

It then follows from (1.3) that

$$\psi_{r,s}(h,k)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}B_j\,B_{r+s-j}\quad(r+s\text{ odd}).\tag{4.6}$$

In particular $\phi_{r,s}(h,k)$, $\psi_{r,s}(h,k)$ are independent of h, k when $r+s$ is odd.

5. We shall require the following

LEMMA. *Let a, b, c be positive integers such that*

$$(b,c)=(c,a)=(a,b)=1.\tag{5.1}$$

Then we have the following identity:

$$\left\{\begin{aligned}&ac\sum_{\eta}\frac{u^c\eta^{c-a}}{(u^a-\eta^{-a})(v^c-u^c\eta^c)}+ab\sum_{\zeta}\frac{v^b\zeta^{b-a}}{(v^a-\zeta^{-a})(u^b-v^b\zeta^b)}\\&\qquad=bc\sum_{\zeta}\frac{\xi^{b+c}}{(u^b-\xi^b)(v^c-\xi^c)},\end{aligned}\right.\tag{5.2}$$

where ξ, η, ζ run through the roots of unity of index a, b, c, respectively.

PROOF. The proof is similar to that of (2.1). It follows from

$$\frac{au^a}{u^a-v^a}=\sum_{\zeta}\frac{u}{u-v\xi}\tag{5.3}$$

that

$$ac\sum_{\eta}\frac{u^c\eta^{c-a}}{(u^a-\eta^{-a})(u^c\eta^c-v^c)}=a\sum_{\eta,\zeta}\frac{u\eta^{-a+1}}{(u^a-\eta^{-a})(u\eta-v\zeta)}$$

$$=-\sum_{\eta,\zeta}\frac{u\eta}{u\eta-v\zeta}\,\frac{a\eta^{-a}}{\eta^{-a}-u^a}=\sum_{\xi,\eta,\zeta}\frac{u}{(u\eta-v\zeta)(u\xi-\eta^{-1})}.$$

Similarly

$$ab\sum_{\zeta}\frac{v^b\zeta^{b-a}}{(v^a-\zeta^{-a})(v^b\zeta^b-u^b)}=\sum_{\xi,\eta,\lambda}\frac{v}{(v\zeta-u\eta)(v\xi-\zeta^{-1})}.$$

Hence the left member of (5.2) is equal to

$$-\sum_{\xi,\eta,\xi}\frac{1}{u\eta-v\zeta}\left(\frac{u}{u\xi-\eta^{-1}}-\frac{v}{v\xi-\zeta^{-1}}\right)=-\sum_{\xi,\eta,\eta}\frac{1}{u\eta-v\zeta}\,\frac{v\eta^{-1}-u\zeta^{-1}}{(u\xi-\eta^{-1})(v\xi-\zeta^{-1})}$$

$$=\sum_{\xi,\eta,\zeta}\frac{\eta^{-1}\zeta^{-1}}{(u\xi-\eta^{-1})(v\xi-\zeta^{-1})}=\sum_{\xi}\sum_{\eta}\frac{\eta}{u\xi-\eta}\sum_{\zeta}\frac{\zeta}{v\xi-\zeta}$$

$$=bc\sum_{\xi}\frac{1}{u^b\xi^b-1}\,\frac{1}{v^c\xi^c-1}=bc\sum_{\xi}\frac{\xi^{b+c}}{(u^b-\xi^b)(v^c-\xi^c)}.$$

This completes the proof of the Lemma.

Note that, when $a=1$, (5.2) reduces to (2.1).

6. In the remainder of the paper we assume that a, b, c satisfy (5.1). It will be convenient to introduce the following notation:

$$\phi_{r,s}(b,c;a)=\sum_{t\,(\mathrm{mod}\,a)}\bar{B}_r\left(\frac{bt}{a}\right)\bar{B}_s\left(\frac{ct}{a}\right), \tag{6.1}$$

so that

$$\phi_{r,s}(b,c;a)=\phi_{r,s}(bc',a), \tag{6.2}$$

where $cc'\equiv 1 \pmod a$. We also define

$$\psi_{r,s}(b,c;a)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}b^{r-j}c^{j-r}\phi_{j,r+s-j}(b,c;a). \tag{6.3}$$

Put

$$\Phi(b,c;a|x,y)=\sum_{r,s=0}^{\infty}\phi_{r,s}(b,c;a)\frac{x^r y^s}{r!s!}, \tag{6.4}$$

$$\Psi(b,c;a|x,y)=\sum_{r,s=0}^{\infty}\psi_{r,s}(b,c;a)\frac{x^r y^s}{r!s!}. \tag{6.5}$$

It is clear from (6.2) that

$$\Phi(b,c;a|x,y)=\Phi(bc',a|x,y)=\Phi(c,b;a|y,x). \tag{6.6}$$

It is easily proved, using (6.3), that

$$\Psi(b,c;a|x,y)=\Phi\left(b,c;a|x,y-\frac{bx}{c}\right). \tag{6.7}$$

By (3.7) and (6.6),

$$\Phi(b,a;c|cy,cx)=ax\sum_{\zeta}\frac{\zeta^{ba'}-1}{(e^y-\zeta^{-1})(e^x-\xi^{ba'})} \tag{6.8}$$

where ζ runs through the roots of unity of index c. Replacing ξ by ξ^a, (6.8) becomes

$$\Phi(b,a;c|cy,cx)=cxy\sum_{\zeta}\frac{\zeta^{b-a}}{(e^y-\zeta^{-a})(e^x-\zeta^{b})}. \tag{6.9}$$

Hence, by (6.7), we get

$$\Psi(b,a;c|cy,cx)=\frac{cy(ax-by)}{a}\sum_{\zeta}\frac{\zeta^{b-a}}{(e^y-\zeta^{-a})(e^{x-by/a}-\zeta^{b})}. \tag{6.10}$$

Similarly

$$\Psi(c,a;b|bx,by)=\frac{bx(ay-cx)}{a}\sum_{\eta}\frac{\eta^{c-a}}{(e^x-\eta^{-a})(e^{y-cx/a}-\eta^{c})}. \tag{6.11}$$

It follows from (6.10) and (6.11) that

$$\Psi(b, a; c|acy, bcx) = abcy(y-x) \sum_{\zeta} \frac{\zeta^{b-a}}{(e^{ay}-\zeta^{-a})(e^{bx-by}-\zeta^{b})},$$

$$\Psi(c, a; b|abx, bcx) = abcx(x-y) \sum_{\eta} \frac{\eta^{c-a}}{(e^{ax}-\eta^{-a})(e^{cy-cx}-\eta^{b})},$$

so that

$$(6.12)\quad \left\{\begin{aligned} & bx\Psi(b, a; c|acy, bcx) - cy\Psi(c, a; b|abx, bcy) \\ & \quad = bcxy(x-y)\left\{ac \sum_{\eta} \frac{\eta^{c-a}}{(e^{ax}-\eta^{-a})(e^{cy-cx}-\eta^{c})}\right. \\ & \quad \left. + ab \sum_{\xi} \frac{\zeta^{b-a}}{(e^{ay}-\zeta^{-a})(e^{bx-by}-\zeta^{b})}\right\}. \end{aligned}\right.$$

By (6.8) we have

$$\Phi(c, -b; a|abx, acy) = abcxy \sum_{\xi} \frac{\xi^{b+c}}{(e^{bx}-\xi^{b})(e^{cy}-\xi^{c})}.$$

It follows that

$$abx\Psi(b, a; c|acy, bcx) - acy\Psi(c, a; b|abx, bcy) - bc(x-y)\Phi(c, -b; a|abx, acy)$$

$$= abcxy(x-y)\left\{ac \sum_{\eta} \frac{\eta^{c-a}}{(e^{ax}-\eta^{-a})(e^{cy-cx}-\eta^{c})} + ab \sum_{\zeta} \frac{\zeta^{b-a}}{(e^{ay}-\zeta^{-a})(e^{bx-by}-\zeta^{b})}\right.$$

$$\left. - bc \sum_{\xi} \frac{\xi^{b+c}}{(e^{bx}-\xi^{b})(e^{by}-\xi^{c})}\right\}.$$

In (5.2) replace u, v by e^x, e^y, respectively. Clearly the quantity within braces above vanishes and we therefore have

$$(6.13)\quad \left\{\begin{aligned} & abx\Psi(b, a; c|acy, bcx) - acy\Psi(c, a; b|abx, bcy) \\ & = bc(x-y)\Phi(c, -b; a|abx, acy). \end{aligned}\right.$$

7. In view of (6.4) and (6.5), (6.13) becomes

$$abx \sum_{r,s=0}^{\infty} \psi_{s,r}(b, a; c) \frac{(bcx)^r(acy)^s}{r!\,s!} - acy \sum_{r,s=0}^{\infty} \psi_{r,s}(c, a; b) \frac{(abx)^r(bcy)^s}{r!\,s!}$$

$$= bc(x-y) \sum_{r,s=0}^{\infty} \phi_{r,s}(c, -b; a) \frac{(abx)^r(acy)^s}{r!\,s!}.$$

Comparing coefficients of $x^r y^s$ on both sides we get

$$(7.7)\quad \left\{\begin{aligned} & ra^{s+1}c^{r-1}\psi_{s,r-1}(b, a; c) - sa^{r+1}b^{s-1}\psi_{r,s-1}(c, a; b) \\ & \qquad = ra^{r+s}c\phi_{r-1,s}(-b'c, a) - sa^{r+s}b\phi_{r,s-1}(-b'c, a), \end{aligned}\right.$$

where $bb' \equiv 1 \pmod{a}$.

We may now state

THEOREM 2. *Let a, b, c be positive integers such that*

$$(b, c) = (c, a) = (a, b) = 1.$$

Then the relation (7.1) *holds for all non-negative r, s.*

It is evident from (6.1) that

$$\phi_{r,s}(b, c; 1) = B_r B_s$$

for arbitrary b, c. Hence, for $a = 1$, (7.1) reduces to

$$rc^{r-1}\psi_{s,r-1}(b, c) - sb^{s-1}\psi_{r,s-1}(c, b) = rcB_{r-1}B_s - sbB_rB_{s-1},$$

in agreement with (4.4).

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